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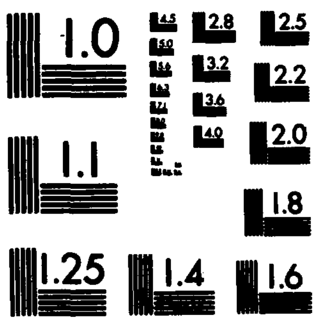

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A MULTIVARIATE IFR CLASS

by

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ABSTRACT

A nonnegative random vector $\underset{\sim}{T}$ is said to have a multivariate increasing failure rate distribution (MIFR) if and only if $E[h(\underset{\sim}{x},\underset{\sim}{T})]$ is log concave in $\underset{\sim}{x}$ for all functions $h(\underset{\sim}{x},\underset{\sim}{t})$ which are log concave in $(\underset{\sim}{x},\underset{\sim}{t})$ and are nondecreasing and continuous in $\underset{\sim}{t}$ for each fixed $\underset{\sim}{x}$. This class of distributions is closed under deletion, conjunction, convolution and weak limits. It contains the multivariate exponential distribution of Marshall and Olkin and those distributions having a log concave density. Also, it follows that if $\underset{\sim}{T}$ is MIFR and ψ is nondecreasing, nonnegative and concave then $\psi(\underset{\sim}{T})$ is IFR.

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Key words: multivariate, IFR, log concave, inequalities.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A nonnegative random vector $\underset{\sim}{T}$ is said to have a multivariate increasing failure rate distribution (MIFR) if and only if $E[h(\underset{\sim}{x},\underset{\sim}{T})]$ is log concave in $\underset{\sim}{x}$ for all functions $h(\underset{\sim}{x},\underset{\sim}{t})$ which are log concave in $(\underset{\sim}{x},\underset{\sim}{t})$ and are nondecreasing and continuous in $\underset{\sim}{t}$ for each fixed $\underset{\sim}{x}$. This class of distributions is closed under deletion, conjunction, convolution and weak limits. It contains the multivariate exponential distribution of Marshall and Olkin and those distribu-

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§6. Relation to the MIFRA class.

In Block and Savits (1980), we defined a multivariate increasing failure rate average (MIFRA) class as follows. A nonnegative random vector $\underset{\sim}{T}$ is said to be MIFRA if $E[h(\underset{\sim}{T})] \leq E^{1/\alpha}[h^{\alpha}(\underset{\sim}{T}/\alpha)]$ for all $0<\alpha<1$ and all nonnegative nondecreasing functions h; equivalently, $P(\underset{\sim}{T} \in A) \leq P^{1/\alpha}(\underset{\sim}{T} \in \alpha A)$ for all $0<\alpha<1$ and all upper sets A. It is well known in the univariate case that if T is IFR, then T is IFRA (see Barlow and Proschan (1975)). It seems natural to expect the same conclusion in the multivariate setting. Although we believe this to be the case, we only have some partial results.

Let $\underset{\sim}{T}$ be MIFR and μ its induced measure. Let A be an upper convex set. Then for $\alpha>0$, $h(\alpha,\underset{\sim}{t}) = I_A(\underset{\sim}{t}/\alpha)$ is log concave in $(\alpha,\underset{\sim}{t})$ and nondecreasing in $\underset{\sim}{t}$ for each fixed α. We may assume without loss of generality that $P(\underset{\sim}{T}>\underset{\sim}{0})=1$; otherwise $P(T_i=0)>0$ for some i. But T_i is IFR and so $P(T_i=0)=1$. We are thus reduced to the case of one less dimension. Since $\underset{\sim}{T}$ is MIFR, it follows that $G(\alpha)=E[h(\alpha,\underset{\sim}{T})]$ is log concave in $\alpha>0$. Also, since $P(\underset{\sim}{T}>0)=1$, we get that $G(0+)=1$. Consequently, we conclude that $-\log G(\alpha)$ is star-shaped and so $\mu(\alpha A) \geq \mu^{\alpha}(A)$.

This result coupled with the fact that every upper set can be approximated by a finite union of upper convex sets leads us to the following conjecture.

(6.1) Conjecture. If $\underset{\sim}{T}$ is MIFR, then $\underset{\sim}{T}$ is MIFRA.

Now let $\varepsilon \downarrow 0$.

(5.7) Corollary. A nonnegative random vector $\underset{\sim}{T}$ is MIFR if and only if $E[h(\underset{\sim}{x},\underset{\sim}{T})]$ is log concave in $\underset{\sim}{x}$ for all functions h which are log concave in $(\underset{\sim}{x},\underset{\sim}{t})$ and are nondecreasing in $\underset{\sim}{t}$ for each fixed $\underset{\sim}{x}$.

fixed λ. Let $h(\lambda,\underset{\sim}{y}) = \exp\{-\phi(\lambda,\underset{\sim}{y})\}$. Then h is log concave in $(\lambda,\underset{\sim}{y})$, and continuous and nondecreasing in $\underset{\sim}{y}$ for each fixed λ. Consequently, $h_n(\lambda,\underset{\sim}{y}) = h^n(\lambda,\underset{\sim}{y})$ has the same properties for each n. Using the fact that $\underset{\sim}{T}$ is MIFR and that $h_n(\lambda,\underset{\sim}{y}) \to I_D(\lambda,\underset{\sim}{y})$ as $n \to \infty$, we obtain the inequality (2.4).

(5.5) Lemma. Let μ be a probability measure and A an upper convex set. Then given $\varepsilon > 0$ there exists a closed upper convex set $C \subset A$ such that $\mu(C) \geq \mu(A) - \varepsilon$.

Proof. Given $\varepsilon > 0$, choose a compact set $K \subset A$ such that $\mu(K) \geq \mu(A) - \varepsilon$. If G is the convex hull of K, then G is a compact convex set satisfying $K \subset G \subset A$. As in Block and Savits (1980), define $C = \{\underset{\sim}{y} + \underset{\sim}{t}: \underset{\sim}{y} \in G, \underset{\sim}{t} \geq \underset{\sim}{0}\}$. It is not hard to show that C has the desired properties; i.e., C is a closed upper convex set satisfying $K \subset G \subset C \subset A$. Consequently $\mu(C) \geq \mu(A) - \varepsilon$.

(5.6) Theorem. If μ is a probability measure satisfying the inequality $\mu[\lambda A + (1-\lambda)B] \geq \mu^{\lambda}(A)\mu^{1-\lambda}(B)$ for all $0 < \lambda < 1$ and all closed upper convex sets A,B, then the inequality remains valid for all upper convex sets A,B.

Proof. It suffices to consider upper convex sets A and B having positive μ measure. Given $\varepsilon > 0$, sufficiently small, we choose closed upper convex sets $C \subset A$, $D \subset B$ as in Lemma 5.5 such that $\mu(C) \geq \mu(A) - \varepsilon$, $\mu(D) \geq \mu(B) - \varepsilon$. Hence

$$\begin{aligned}\mu[\lambda A + (1-\lambda)B] &\geq \mu[\lambda C + (1-\lambda)D] \\ &\geq \mu^{\lambda}(C)\mu^{1-\lambda}(D) \geq [\mu(A)-\varepsilon]^{\lambda}[\mu(B)-\varepsilon]^{1-\lambda}.\end{aligned}$$

$$h[\lambda x + (1-\lambda)x', \ \lambda w + (1-\lambda)w'] = \mu\{D[\lambda x + (1-\lambda)x', \ \lambda w + (1-\lambda)w']\}$$

$$\geq \mu\{\lambda D(x,w) + (1-\lambda)D(x',w')\}$$

$$\geq \mu^{\lambda}[D(x,w)]\mu^{1-\lambda}[D(x',w')] = h^{\lambda}(x,w)h^{1-\lambda}(x',w');$$

i.e., h is log concave in (x,w). Thus by Theorem 2.4, G is log concave in x.

We now consider the converse. Let A and B be upper convex sets. For $0 \leq \lambda \leq 1$, let $C_\lambda = \lambda A + (1-\lambda)B$. We define $D = \{(\lambda,y): 0 \leq \lambda \leq 1 \text{ and } y \in C_\lambda\}$. Then D is convex and so $h(\lambda,y) = I_D(\lambda,y)$ is log concave. It is not hard to show that h is nondecreasing in y for each fixed λ. By hypothesis, then, $g(\lambda) = \int h(\lambda,y)\mu(dy)$ is log concave in λ. Hence

$$\mu[\lambda A + (1-\lambda)B] = g(\lambda) \geq g^{\lambda}(1)g^{1-\lambda}(0) = \mu^{\lambda}(A)\mu^{1-\lambda}(B).$$

(5.3) <u>Theorem</u>. Let T be a nonnegative random vector and $\mu(dy) = P(T \in dy)$ be its induced measure. Then T is MIFR if and only if

$$\mu[\lambda A + (1-\lambda)B] \geq \mu^{\lambda}(A)\mu^{1-\lambda}(B) \tag{5.4}$$

for all $0 < \lambda < 1$, all closed upper convex sets A,B.

<u>Proof</u>. The same proof as in Theorem 5.1 shows that if (5.4) holds then T is MIFR; if $F(x,y)$ is also continuous in y for each fixed x, then the sets $\{y: F(x,y) \geq z\}$ are also closed.

Now let A,B be closed upper convex sets. As before, let $C_\lambda = \lambda A + (1-\lambda)B$ for $0 \leq \lambda \leq 1$ and set $D = \{(\lambda,y): 0 \leq \lambda \leq 1, \ y \in C_\lambda\}$. Now D is a closed convex set. We set $\phi(\lambda,y) = \rho(\lambda,y),D)$ where ρ is the metric $\rho(u,v) = \max |u_i - v_i|$. Note that ϕ is continuous and convex. It is also a nonincreasing function in y for each

§5. An alternative condition.

In this section (which is somewhat technical) we derive an alternative condition that T be MIFR; it is directly expressable in terms of the induced measure $\mu(dy) = P(T \in dy)$. This result, coupled with some approximation ideas, allows us to remove the continuity assumption on h in Definition 4.1.

We first recall that a set $A \subset \mathbb{R}^n$ is said to be an upper set if whenever $x \in A$ and $y \geq x$, then $y \in A$.

(5.1) Theorem. Let μ be a finite measure. In order that $\int F(x,y)\mu(dy)$ be log concave in x for all log concave functions $F(x,y)$ which are nondecreasing in y for each fixed x it is necessary and sufficient that

$$\mu[\lambda A + (1-\lambda)B] \geq \mu^{\lambda}(A)\mu^{1-\lambda}(B) \tag{5.2}$$

for all $0<\lambda<1$ and all upper convex sets A,B.

Proof. We shall closely follow the argument given in Brascamp and Lieb (1975). Suppose now that μ satisfies (5.2) and let $F(x,y)$ satisfy the conditions of the theorem. For each real $z \geq 0$, let $C(x,z) = \{y: F(x,y) \geq z\}$ and set $g(x,z) = \mu[C(x,z)]$. Note that $C(x,z)$ is upper and convex; furthermore, $G(x) = \int F(x,y)\mu(dy) = \int_0^\infty g(x,z)dz$. If we make the change of variables $z = e^w$, then $G(x) = \int_{-\infty}^{\infty} h(x,w)e^w dw$ where $h(x,w) = g(x,e^w) = \mu[C(x,e^w)]$. We also define $D(x,w) = C(x,e^w)$. It is easy to show that $D[\lambda x + (1-\lambda)x', \lambda w + (1-\lambda)w'] \supset \lambda D(x,w) + (1-\lambda)D(x',w')$ for all $0<\lambda<1$, x,x',w,w'. Consequently, assuming (5.2), we obtain the inequality

ponentially distributed random variables $S_1,\ldots,S_n$ and subsets $J_1,\ldots,J_m$ of $\{1,\ldots,n\}$ such that $T_i = \min_{j\in J_i} S_j$. But $(S_1,\ldots,S_n)$ is MIFR and $\psi_i(\underset{\sim}{s}) = \min_{j\in J_i} s_j$ $(1\leq i\leq m)$ satisfy the hypothesis of Theorem 4.3(v).

(4.6) Example 2. If $\underset{\sim}{T}$ has a density which is log concave, then $\underset{\sim}{T}$ is MIFR. Apply Theorem 2.4.

$S_i = \psi_i(\mathbf{T})$, $1 \le i \le m$, with $\mathbf{T}$ MIFR, then $E[h(\mathbf{x},\mathbf{S})] = E[g(\mathbf{x},\mathbf{T})]$ is log concave in $\mathbf{x}$.

(ii) Let $h(\mathbf{x},\mathbf{s},\mathbf{t})$ be log concave in $(\mathbf{x},\mathbf{s},\mathbf{t})$ and continuous and nondecreasing in $(\mathbf{s},\mathbf{t})$ for each fixed $\mathbf{x}$. We first assume that h is bounded. Since $\mathbf{T}$ is MIFR it follows that $g(\mathbf{x},\mathbf{s}) = E[h(\mathbf{x},\mathbf{s},\mathbf{T})]$ is log concave in $(\mathbf{x},\mathbf{s})$. Also, g is continuous and nondecreasing in $\mathbf{s}$ for each fixed $\mathbf{x}$. Thus, since $\mathbf{S}$ is MIFR, $E[g(\mathbf{x},\mathbf{S})]$ is log concave in $\mathbf{x}$. But, by Fubini, $E[g(\mathbf{x},\mathbf{S})] = E[h(\mathbf{x},\mathbf{s},\mathbf{T})]$. If h is not bounded, consider instead $h \wedge n$ and pass to the limit (see Theorem 2.3(i)).

(i) If $J \subset \{1,\ldots,n\}$, let $\psi_j(\mathbf{t}) = t_j$ for $j \in J$. According to Theorem 4.3(v), $\{\psi_j(\mathbf{T}) = T_j;\ j \in J\}$ is MIFR.

(iii) Since $\mathbf{S}$ and $\mathbf{T}$ are independent MIFR, $(\mathbf{S},\mathbf{T})$ is MIFR by Theorem (4.3(ii). Now take $\psi_j(\mathbf{s},\mathbf{t}) = s_j + t_j$ and use Theorem 4.3(v).

(iv) Use Theorem 4.3(v) with $\psi_j(\mathbf{t}) = a_j t_j$.

(v) Let $h(\mathbf{x},\mathbf{t})$ be a bounded log concave function in $(\mathbf{x},\mathbf{t})$ which is nondecreasing and continuous in $\mathbf{t}$ for each fixed $\mathbf{x}$. Since $\mathbf{T}_n \to \mathbf{T}$ in distribution, $E[h(\mathbf{x},\mathbf{T}_n)] \to E[h(\mathbf{x},\mathbf{T})]$ as $n \to \infty$. But for each n, $E[h(\mathbf{x},\mathbf{T}_n)]$ is log concave in $\mathbf{x}$. Consequently, $E[h(\mathbf{x},\mathbf{T})]$ is log concave in $\mathbf{x}$. If h is not bounded, use the argument as in the proof of part (ii).

(4.4) <u>Corollary</u>. If $T_1,\ldots,T_n$ are independent IFR random variables and $\psi(t_1,\ldots,t_n)$ is continuous, nonnegative, nondecreasing and concave, then $\psi(T_1,\ldots,T_n)$ is IFR.

(4.5) <u>Example 1</u>. The Marshall and Olkin MVE (1967) distribution is MIFR. This follows since if $\mathbf{T} = (T_1,\ldots,T_m)$ is MVE then there exist independent ex-

§4. A multivariate IFR class.

(4.1) Definition. Let $\underset{\sim}{T}$ be a nonnegative random vector. We say that $\underset{\sim}{T}$ has a multivariate increasing failure rate (MIFR) distribution if and only if $E[h(\underset{\sim}{x},\underset{\sim}{T})]$ is log concave in $\underset{\sim}{x}$ for all functions $h(\underset{\sim}{x},\underset{\sim}{t})$ which are log concave in $(\underset{\sim}{x},\underset{\sim}{t})$ and nondecreasing and continuous in $\underset{\sim}{t} \geq \underset{\sim}{0}$ for each fixed $\underset{\sim}{x} \geq \underset{\sim}{0}$.

(4.2) Remarks.

(i) Again note that according to Theorem 2.3(iii) we need only consider functions $h(x,\underset{\sim}{t})$ with x a single variable instead of a vector.

(ii) In Section 5 we shall show that the continuity assumption is unnecessary.

The class of MIFR distributions has many desirable closure properties.

(4.3) Theorem.

(i) If $\underset{\sim}{T}$ is MIFR, then so are all marginals.

(ii) If $\underset{\sim}{S}$ and $\underset{\sim}{T}$ are independent MIFR, then $(\underset{\sim}{S},\underset{\sim}{T})$ is MIFR.

(iii) If $\underset{\sim}{S}$ and $\underset{\sim}{T}$ are independent MIFR of the same dimension, then $\underset{\sim}{S}+\underset{\sim}{T}$ is MIFR.

(iv) If $(T_1,\ldots,T_n)$ is MIFR and $a_i \geq 0$ $(i=1,\ldots,n)$, then $(a_1T_1,\ldots,a_nT_n)$ is MIFR.

(v) If $\underset{\sim}{T}$ is MIFR and $\psi_1,\ldots,\psi_m$ are continuous, nonnegative, nondecreasing and concave functions, then $(\psi_1(\underset{\sim}{T}),\ldots,\psi_m(\underset{\sim}{T}))$ is MIFR.

(vi) If $\underset{\sim}{T}_1,\underset{\sim}{T}_2,\ldots$ are MIFR and $\underset{\sim}{T}_n$ converges to $\underset{\sim}{T}$ is distribution, then $\underset{\sim}{T}$ is MIFR.

Proof.

(v) Let $h(\underset{\sim}{x},\underset{\sim}{s})$ be log concave in $(\underset{\sim}{x},\underset{\sim}{s})$ and continuous, and nondecreasing in $\underset{\sim}{s}$ for each fixed $\underset{\sim}{x}$. If $\psi_1,\ldots,\psi_m$ are as in (v), then $g(\underset{\sim}{x},\underset{\sim}{t}) = h(\underset{\sim}{x},\psi_1(\underset{\sim}{t}),\ldots,\psi_m(\underset{\sim}{t}))$ has the same properties as h (see Theorem 2.3(ii)). Consequently, if

Proof. Let $R(t) = -\log \bar{F}(t)$ be the hazard function of T. We set $a = \sup\{t \geq 0: \bar{F}(t) = 1\}$ and $b = \inf\{t \geq 0: \bar{F}(t) = 0\}$ ($\inf \emptyset = +\infty$). If $a = b$, we simply take $\psi \equiv a$ and we are done; otherwise $0 \leq a < b \leq +\infty$. Since R is convex and finite on $(-\infty, b)$ it is continuous there; furthermore, it easily follows that R is strictly increasing on $[a,b)$. Let $A = \lim_{t \to b-} R(t) \leq +\infty$. If ϕ denotes the restriction of R to $[a,b)$, then its inverse ϕ^{-1} is continuous, strictly increasing and concave on $[0,A)$. The function ψ is defined by $\psi(s) = \inf\{t \geq 0: R(t) > s\}$. Clearly $\psi(s) = \phi^{-1}(s)$ for $0 \leq s < A$ and $\psi(s) = b$ for $s \geq A$. It is not hard to show now that ψ has the desired properties.

(3.4) Theorem. T is IFR if and only if $E[h(\underset{\sim}{x},T)]$ is log concave in $\underset{\sim}{x}$ for all functions $h(\underset{\sim}{x},t)$ which are log concave in $(\underset{\sim}{x},t)$ and are nondecreasing in t for each fixed $\underset{\sim}{x} \geq \underset{\sim}{0}$.

Proof. Suppose T is IFR and let h be as in the statement of the theorem. Then, according to Lemma 3.3, there exists a continuous nonnegative nondecreasing concave function ψ such that $\psi(S)$ and T have the same distribution, where S has the standard exponential distribution. Hence

$$E[h(\underset{\sim}{x},T)] = E[h(\underset{\sim}{x},\psi(S))] = \int_0^\infty h(\underset{\sim}{x},\psi(s))e^{-s}ds$$

is log concave in $\underset{\sim}{x}$. This follows from Theorems 2.3 (i),(ii) and 2.4.

To prove the converse, let $h(x,t) = I_{(x,\infty)}(t)$. Then h is log concave in (x,t) and nondecreasing in t for each fixed x. By assumption, then, $\bar{F}(x) = E[h(x,T)]$ is log concave in x. According to Theorem 3.1, T is IFR.

(3.5) Remark. According to Theorem 2.3(iii), we need only consider functions $h(x,t)$ with x a single variable instead of a vector.

§3. A new univariate IFR characterization.

The univariate class of increasing faliure rate (IFR) distributions has played an important role in the mathematical theory of reliability (see Barlow and Proschan (1975)). We shall first briefly review the definition. Let T be a nonnegative random variable with survival probability $\bar{F}(t) = P(T > t)$. Set $b = \inf\{t \geq 0: \bar{F}(t) = 0\}$ ($\inf \emptyset = +\infty$). We say that T has an IFR distribution if $\bar{F}(s+t)/\bar{F}(t)$ is nonincreasing in $t \in [0,b)$ for each $s \geq 0$. It is well known that the following conditions are equivalent (cf. Barlow and Proschan (1975)).

(3.1) Theorem. The following conditions are equivalent:

(i) T is IFR.

(ii) $\bar{F}$ is a Polya frequency function of order two (PF_2), i.e., $\bar{F} \geq 0$ and

$$\begin{vmatrix} \bar{F}(x_1-y_1) & \bar{F}(x_1-y_2) \\ \bar{F}(x_2-y_1) & \bar{F}(x_2-y_2) \end{vmatrix} \geq 0$$

for all $-\infty < x_1 < x_2 < \infty$, $-\infty < y_1 < y_2 < \infty$.

(iii) $\bar{F}$ is log concave.

(3.2) Remark. If F has a density f, then T is IFR if and only if the hazard rate $r(t) = f(t)/\bar{F}(t)$ is nondecreasing on $[0,b)$.

Before we state our new characterization, we need the following simple result.

(3.3) Lemma. If T is IFR, then there exists a continuous nonnegative nondecreasing concave function ψ on $[0,\infty)$ such that $\psi(S)$ has the same distribution as T, where S is distributed as a standard exponential.

The following facts about log concave functions are easily verifiable.

(2.3) Theorem.

(i) If f and g are log concave, so are $f\cdot g$, $f\wedge g$, cf and f^a for all $a > 0$, $c \geq 0$.

(ii) If f is log concave and nondecreasing and ψ is concave (and nonnegative), then the composition $f \circ \psi$ is log concave.

(iii) f is log concave if and only if for every $\underset{\sim}{x}, \underset{\sim}{y} \geq \underset{\sim}{0}$, the function $g(t) = f[t\underset{\sim}{x} + (1-t)\underset{\sim}{y}]$ is log concave on $0 \leq t \leq 1$.

(iv) If f is log concave, then for every z, the sets $\{f > z\}$ and $\{f \geq z\}$ are convex.

(v) If f is log concave, then f is continuous on the interior of the set $\{f > 0\}$.

We close this section with a very important result about log concave functions. This result is sometimes known as the Prékopa Theorem (1971). An independent and simpler proof is given in Brascamp and Lieb (1975).

(2.4) Theorem. Let F be log concave on $\mathbb{R}^m \times \mathbb{R}^n$. Then $G(\underset{\sim}{x}) = \int F(\underset{\sim}{x},\underset{\sim}{y})d\underset{\sim}{y}$ is log concave on $\mathbb{R}^m$. (Here $d\underset{\sim}{y}$ is Lebesgue measure on $\mathbb{R}^n$).

The above plays a crucial role in our development.

§2. Log concave functions.

Let A be a convex set in $\mathbb{R}^n$ and f a nonnegative function defined on A. We say that f is log concave on A if

$$f[\lambda \underset{\sim}{x} + (1-\lambda)\underset{\sim}{y}] \geq f^{\lambda}(\underset{\sim}{x}) f^{1-\lambda}(\underset{\sim}{y}) \tag{2.1}$$

for all $0 < \lambda < 1$, all $\underset{\sim}{x}, \underset{\sim}{y} \in A$. Sometimes we write $f(\underset{\sim}{x}) = e^{-Q(\underset{\sim}{x})}$ where $Q(\underset{\sim}{x})$ is convex, but with the understanding that Q may assume the value $+\infty$.

(2.2) Examples

(i) If $Q(\underset{\sim}{x})$ is twice continuously differentiable on an open convex set A and has the property that at each point $\underset{\sim}{x} \in A$, the matrix $\left[\frac{\partial^2 Q(\underset{\sim}{x})}{\partial x_i \partial x_j}\right]$ is nonnegative definite, then $f(\underset{\sim}{x}) = e^{-Q(\underset{\sim}{x})}$ is log concave on A. Hence all Gaussian densities are log concave.

(ii) If A is any convex set in $\mathbb{R}^n$, then the indicator function $I_A(\underset{\sim}{x})$ is log concave on $\mathbb{R}^n$.

(iii) If f is a nonnegative concave function, then f is log concave.

It is convenient to make the following simple observation. Let A and B be convex sets in $\mathbb{R}^n$ and suppose that f is log concave on A. Clearly then, f is log concave on B if $B \subset A$. On the other hand suppose $A \subset B$. Then the function $\tilde{f}$, which is defined to be equal to f on A and zero on $B \backslash A$, is log concave on B. Thus without loss of generality we may assume that all log concave functions are defined on the same convex set. In the context of this paper, it is natural to work with the convex set $\mathbb{R}^n_+$. Hence, unless otherwise specified, the term log concave means log concave on $\mathbb{R}^n_+$.

§1. Introduction.

Various univariate classes of life distributions have been introduced in the mathematical theory of reliability (see Barlow and Proschan (1975) for a detailed discussion). Recently there has been much interest in obtaining multivariate versions of these classes. Although there have been many different approaches, this document mentions ~~(see, e.g., the review paper by Block and Savits (1981), we mention~~ only two because of their nice closure properties: the multivariate IFRA class of Block and Savits (1980) and the multivariate NBU class of Marshall and Shaked (1982). Both of these classes are closed under deletion, conjunction, convolution and weak limits.

This paper introduces ~~In this paper we introduce~~ a multivariate IFR class that has similar closure properties. Its spirit also closely parallels that found in the two previously mentioned papers. Other definitions of a multivariate IFR class have been proposed by Marshall (1975), but none possess all desirable closure properties.

~~In~~ Section 2 presents ~~we present~~ some preliminary facts about log concave functions. Section 3 contains a new characterization of the (univariate) IFR class. The multivariate generalization and properties thereof are given in Section 4. A useful alternative condition is delineated in Section 5. Finally, ~~we compare~~ this class is compared with the multivariate IFRA class of Block and Savits (1980) in Section 6.

All functions and sets in this paper are assumed to be Borel measureable with respect to $\mathbb{R}^n$. In most cases, however, we do not specify the dimension n directly; it is usually clear from the context.